Stochastic Domination of Exit Times for Random Walks and Brownian Motion with Drift

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Abstract

In this note, by an elementary use of Girsanov's transform we show that the exit time for either a biased random walk or a drifted Brownian motion on a symmetric interval is stochastically monotone with respect to the drift parameter. In the random walk case, this gives an alternative proof of a recent result of E. Peköz and R. Righter in 2024. Our arguments in both discrete and continuous cases are parallel to each other. We also outline a simple SDE proof for the Brownian case based on a standard comparison theorem.

1 Introduction

In a series of elegant recent papers, several variants of the following problem were addressed: determine the optimal drift for 1-dimensional simple random walk or Brownian motion starting at 0 to stay for as long as possible in a symmetric

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interval. To be precise, let $\{S^p_n : n \geqslant 0\}$ denote the simple random walk starting at the origin with one-step distribution

$$\mathbb{P}(X=1)=p,\ \mathbb{P}(X=-1)=1-p.$$

This is often referred to as the *biased random walk*. Given $k \in \mathbb{Z}_+$, define

$$\sigma^k_p \triangleq \inf\{n : S^p_n = \pm k\}.$$

In [8, Lemma 2] it was shown that the function $p \to E[\sigma^k_p]$ is increasing for $p \in (0,1/2)$ and decreasing for $p \in (1/2,1)$, thereby showing that this function is maximized at $p=1/2$. Later, in [5, 6], it was shown that the stopping time σ^k_p is stochastically maximized at $p=1/2$, and furthermore if $1/2 \leq p_1 < p_2 \leq 1$ then $\sigma^k_{p_1}$ stochastically dominates $\sigma^k_{p_2}$ (Theorem 1 (i) below). Their proof is based on the use of a clever coupling argument. It was also asserted in [5] that an appeal to Donsker's theorem allows one to deduce the analogous result for Brownian motion (Theorem 1 (ii) below) as a corollary to the result for random walk.

The proofs given in [5, 6, 8] are ingenious but combinatorial in nature, and do not seem easy to be modified in order to prove Theorem 1 (ii) directly (i.e. without proving the result first for simple random walk and then invoking Donsker's theorem). It is natural then to search for a direct proof in the Brownian case. In the end we were able to find two such proofs, one by performing a Girsanov change of measure, and the other by translating the problem into a question about SDE's and applying a standard comparison theorem. Both proofs are relatively short and non-technical, and may be of pedagogical interest for readers unfamiliar with the techniques used. The purpose of this note is to present these two proofs.[1]

The following is the random walk result proved in [5, 6], as well as the analogous Brownian motion result that we will focus on.

Theorem 1. *(i) [Random walk] With notation as above, for any fixed $n \in \mathbb{N}$ the function*

$$(0,1) \ni p \mapsto \mathbb{P}(\sigma^k_p > n)$$

is increasing on $(0,1/2]$ and decreasing on $[1/2,1)$.
(ii) [Brownian motion] Let B_t denote the standard one-dimensional Brownian motion. Given $\lambda \in \mathbb{R}$ and $b>0$, define

$$\tau^b_\lambda \triangleq \inf\{t : B_t + \lambda t \notin (-b,b)\}.$$

[1] In private communication, we were informed by Stephen Muirhead that there was yet a third proof (for the Brownian motion case) based on Anderson's inequality for Gaussian measures (cf. [1, Corollary 3.5]), and using such approach the result could further be generalised to drifted Gaussian processes.

Then for any fixed $t > 0$, the function

$$\lambda \mapsto \mathbb{P}(\tau_\lambda^b > t)$$

is decreasing in λ on $[0, \infty)$.

In the next two sections, we develop the Girsanov proof of the theorem in both the Brownian motion and random walk contexts. Our arguments are entirely parallel in these two cases. In Section 4, we outline the SDE proof for the Brownian case based on a well-known comparison theorem. This SDE proof does not seem to be easily adapted to the discrete setting.

Incidentally, as mentioned earlier, part (*ii*) of the theorem was stated as a corollary to part (*i*) in [5] by invoking the weak convergence of simple random walk to Brownian motion. While this is certainly correct in spirit, it is the opinion of the authors of this paper that making such an argument rigorous would present various unpleasant technicalities. It may be beneficial and simpler to just have a direct proof in the Brownian case, as we have done here.

2 The Girsanov proof

In this section, we present the change-of-measure proof for Brownian motion. As indicated earlier, this proof can be directly adapted, mutatis mutandis, to prove the discrete result as well, and we will discuss this in the next section.

Let us begin by setting up the problem on the canonical sample space. We take Ω to be the continuous path space, $B_t(\omega) \triangleq \omega_t$ to be the coordinate process and $\mathcal{F}_t$ to be the natural filtration of B_t.

Let $\mathbb{Q}_\lambda$ denote the probability measure over Ω under which B_t becomes a Brownian motion with drift λt. Note that $\mathbb{Q}_0$ is simply the standard Wiener measure. We also set $\tau(\omega) \triangleq \inf\{t : B_t(\omega) \notin (-b, b)\}$ ($b > 0$ is given fixed). The following form of Girsanov's Theorem is well-known (see for instance [4, Thm. 10.15] or [3, Thm. 3.5.1]).

Lemma 1. *For each $t > 0$, one has*

$$\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} = e^{\lambda B_t - \frac{1}{2}\lambda^2 t} \quad \text{on } \mathcal{F}_t \quad \text{and} \quad \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} = e^{\lambda B_\tau - \frac{1}{2}\lambda^2 \tau} \quad \text{on } \mathcal{F}_\tau.$$

Furthermore, given two drift values λ_1, λ_2 we have

$$\frac{d\mathbb{Q}_{\lambda_2}}{d\mathbb{Q}_{\lambda_1}} = e^{-(\lambda_1 - \lambda_2)B_t + \frac{\lambda_1^2 - \lambda_2^2}{2} t} \quad \text{on } \mathcal{F}_t \quad \text{and} \quad \frac{d\mathbb{Q}_{\lambda_2}}{d\mathbb{Q}_{\lambda_1}} = e^{-(\lambda_1 - \lambda_2)B_\tau + \frac{\lambda_1^2 - \lambda_2^2}{2} \tau} \quad \text{on } \mathcal{F}_\tau.$$

Proof. The part of the first statement is standard and we omit the proof. The second part of the first statement follows from the fact that if we set $M_t \triangleq e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$, then the stopped martingale $\{M_{\tau\wedge t} : n \geq 0\}$ is bounded and thus uniformly integrable.

The second statement follows from the first by the chain rule. □

Lemma 2. *(cf. [7, P.84]]) Under $\mathbb{Q}_\lambda$, the random variables B_τ and τ are independent.*

Proof. The result is obvious if $\lambda = 0$. For general $H_s \geq 0$, we let f, g be arbitrary test functions. Using Lemma 1 and the independence of B_τ and τ in the $\lambda = 0$ case (several times), one has

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\lambda}\big[f(B_\tau)g(\tau)\big] \\
&= \mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau - \frac{1}{2}\lambda^2\tau} f(B_\tau)g(\tau)\big] = \mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau} f(B_\tau)\big]\mathbb{E}^{\mathbb{Q}_0}\big[e^{-\frac{1}{2}\lambda^2\tau} g(\tau)\big] \\
&= \mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau} f(B_\tau)\big]\mathbb{E}^{\mathbb{Q}_0}\big[e^{-\frac{1}{2}\lambda^2\tau} g(\tau)\big] \times \mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau} e^{-\frac{1}{2}\lambda^2\tau}\big] \quad \text{(the last quantity is just 1)} \\
&= \mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau} e^{-\frac{1}{2}\lambda^2\tau} f(B_\tau)\big]\mathbb{E}^{\mathbb{Q}_0}\big[e^{\lambda B_\tau} e^{-\frac{1}{2}\lambda^2\tau} g(\tau)\big] = \mathbb{E}^{\mathbb{Q}_\lambda}\big[f(B_\tau)\big]\mathbb{E}^{\mathbb{Q}_\lambda}\big[g(\tau)\big].
\end{aligned}$$

The result thus follows. □

We also need the following elementary estimate.

Lemma 3. *Let X be a random variable, and M any real number. Then $\mathbb{E}[X|X \leqslant M] \leqslant \mathbb{E}[X]$.*

Proof. Let ν be the law of X. Then

$$\begin{aligned}
\mathbb{E}[X\mathbf{1}_{\{X\leqslant M\}}] &= \int_{(-\infty,M]} x\nu(dx) = \Big(\int_{(-\infty,M]} x\nu(dx)\Big)\Big(\int_{\mathbb{R}} \nu(dy)\Big) \\
&= \int_{\mathbb{R}^2} x\mathbf{1}_{(-\infty,M]\times\mathbb{R}}(x,y)\nu(dx)\nu(dy)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(X \leqslant M)\cdot\mathbb{E}[X] &= \Big(\int_{(-\infty,M]} \nu(dx)\Big)\Big(\int_{\mathbb{R}} y\nu(dy)\Big) \\
&= \int_{\mathbb{R}^2} y\mathbf{1}_{(-\infty,M]\times\mathbb{R}}(x,y)\nu(dx)\nu(dy).
\end{aligned}$$

The result follows by noting that

$$\int_{(-\infty,M]\times(-\infty,M]} x\nu(dx)\nu(dy) = \int_{(-\infty,M]\times(-\infty,M]} y\nu(dx)\nu(dy)$$

and

$$x\mathbf{1}_{(-\infty,M]\times(M,\infty)}(x,y)\leqslant y\mathbf{1}_{(-\infty,M]\times(M,\infty)}(x,y).$$

□

Proof of Theorem 1 (ii). Let $0\leqslant\lambda_1<\lambda_2$ be given fixed. The starting observation is that

$$\mathbb{Q}_{\lambda_2}(\tau>t)=\mathbb{E}_{\lambda_2}\big[\mathbf{1}_{\{\tau>t\}}\big]=\mathbb{E}_{\lambda_1}\big[e^{-(\lambda_1-\lambda_2)B_\tau+\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\mathbf{1}_{\{\tau>t\}}\big],$$

by Lemma 1. The independence property given by Lemma 2 implies that

$$\begin{aligned}
&\mathbb{E}_{\lambda_1}\big[e^{-(\lambda_1-\lambda_2)B_\tau+\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\mathbf{1}_{\{\tau>t\}}\big]\\
&=\mathbb{E}_{\lambda_1}\big[e^{-(\lambda_1-\lambda_2)B_\tau}\big]\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\mathbf{1}_{\{\tau>t\}}\big]=\frac{\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\mathbf{1}_{\{\tau>t\}}\big]}{\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\big]}\\
&=\frac{\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\big|\tau>t\big]}{\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\big]}\mathbb{Q}_{\lambda_1}(\tau>t)
\end{aligned}$$

It follows that

$$\mathbb{Q}_{\lambda_2}(\tau>t)\leqslant\mathbb{Q}_{\lambda_1}(\tau>t)\iff\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\big|\tau>t\big]\leqslant\mathbb{E}_{\lambda_1}\big[e^{\frac{\lambda_1^2-\lambda_2^2}{2}\tau}\big].$$

The latter inequality is a direct consequence of Lemma 3 since $\lambda_1^2<\lambda_2^2$. □

3 The discrete case

We now discuss the proof of Theorem 1 (i), the discrete case. Let S_n denote unbiased simple random walk with respect to a measure $\mathbb{Q}_{1/2}$; that is, the simple random walk with equal probability of moving to the right and to the left. We need to be able to change measure in order to transform one random walk into another with a different bias. This requires a discrete analog of Lemma 1, i.e. Girsanov's Theorem, and for this we need the discrete analog of the exponential martingale $e^{\lambda B_t-\frac{1}{2}\lambda^2 t}$. This is contained the following lemma.

Lemma 4. *(i) For $p,q\in(0,1)$ with $p+q=1$, the process $M_n=(2\sqrt{pq})^n\Big(\sqrt{\frac{p}{q}}\Big)^{S_n}$ is a martingale with respect to $\mathbb{Q}_{1/2}$.*

(ii) If we define a new measure $\mathbb{Q}_p$ by

$$\frac{d\mathbb{Q}_p}{d\mathbb{Q}_{1/2}} = (2\sqrt{pq})^n \Big(\sqrt{\frac{p}{q}}\Big)^{S_n} \text{ on } \mathcal{F}_n,$$

then with respect to $\mathbb{Q}_p$ *the process* S_n *is biased random walk. To be precise,* $\mathbb{Q}_p(S_{n+1} = r|S_n = r-1) = p$ *and* $\mathbb{Q}_p(S_{n+1} = r|S_n = r+1) = q$*. Furthermore,*

$$\frac{d\mathbb{Q}_p}{d\mathbb{Q}_{1/2}} = (2\sqrt{pq})^\sigma \Big(\sqrt{\frac{p}{q}}\Big)^{S_\sigma} \text{ on } \mathcal{F}_\sigma.$$

(iii) Given two biases p_1, p_2*, we have*

$$\frac{d\mathbb{Q}_{p_2}}{d\mathbb{Q}_{p_1}} = \Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^n \Big(\sqrt{\frac{p_2q_1}{p_1q_2}}\Big)^{S_n} \quad \text{on } \mathcal{F}_n \quad \text{and} \quad \frac{d\mathbb{Q}_{p_2}}{d\mathbb{Q}_{p_1}} = \Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^\sigma \Big(\sqrt{\frac{p_2q_1}{p_1q_2}}\Big)^{S_\sigma} \quad \text{on } \mathcal{F}_\sigma.$$

Proof. For part (i), we note that S_n can be taken as $S_n = \sum_{j=1}^n X_j$, where the X_j's are independent random variables equal to ± 1 with equal probabilities. We note that

$$\mathbb{E}_{\mathbb{Q}_{1/2}}[\Big(\sqrt{\frac{q}{p}}\Big)^{X_j}] = \frac{1}{2}(\sqrt{\frac{q}{p}} + \sqrt{\frac{p}{q}}) = \frac{1}{2\sqrt{pq}}.$$

The process M_n is therefore revealed to be a product martingale with respect to $\mathbb{Q}_{1/2}$.

For part (ii), we note first that part (i) of this lemma was required to show that $\frac{d\mathbb{Q}_p}{d\mathbb{Q}_{1/2}}$ defines a genuine change of measure. We may calculate as follows (assuming $\mathbb{Q}_p(S_n = r-1) > 0$).

$$\mathbb{Q}_p(S_{n+1} = r|S_n = r-1) = \frac{\mathbb{E}_{\mathbb{Q}_p}[1_{\{S_{n+1}=r\}}1_{\{S_n=r-1\}}]}{\mathbb{E}_{\mathbb{Q}_p}[1_{\{S_n=r-1\}}]}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}_{1/2}}\Big[(2\sqrt{pq})^{n+1}\Big(\sqrt{\frac{p}{q}}\Big)^{S_{n+1}} 1_{\{S_{n+1}=r\}}1_{\{S_n=r-1\}}\Big]}{\mathbb{E}_{\mathbb{Q}_{1/2}}\Big[(2\sqrt{pq})^{n}\Big(\sqrt{\frac{p}{q}}\Big)^{S_{n}} 1_{\{S_n=r-1\}}\Big]}$$

$$= \frac{(2\sqrt{pq})^{n+1}\Big(\sqrt{\frac{p}{q}}\Big)^{r}\mathbb{E}_{\mathbb{Q}_{1/2}}\Big[1_{\{S_{n+1}=r\}}1_{\{S_n=r-1\}}\Big]}{(2\sqrt{pq})^{n}\Big(\sqrt{\frac{p}{q}}\Big)^{r-1}\mathbb{E}_{\mathbb{Q}_{1/2}}\Big[1_{\{S_n=r-1\}}\Big]}$$

$$= 2p\mathbb{Q}_{1/2}(S_{n+1} = r|S_n = r-1) = p.$$

A similar calculation shows $\mathbb{Q}_p(S_{n+1} = r | S_n = r+1) = q$. The second part of part (ii) holds because M_n is uniformly integrable.

As in the proof of Lemma 1, part (iii) follows from (ii) by the chain rule. □

Lemma 5. *Under $\mathbb{Q}_p$, the random variables S_σ and σ are independent.*

Proof. The proof is identical, line for line, with that of Lemma 2, provided one replaces B with S, τ with σ, $\mathbb{Q}_\lambda$ with $\mathbb{Q}_p$, and the exponential martingale of Brownian motion with the discrete-time martingale M_n from Lemma 4. □

Proof of Theorem 1 (i). We could argue here, as we have done in the previous lemma, that the proof of this result is the same as for part *(ii)* of this theorem with appropriate substitutions, and with Lemmas 4 and 5 in place of Lemmas 1 and 2 (Lemma 3 needs no modification). However, we include the proof here, for completeness.

Let $1/2 \leqslant p_1 < p_2$ be given fixed. We note that

$$\mathbb{Q}_{p_2}(\sigma > n) = \mathbb{E}_{p_2}\big[\mathbf{1}_{\{\sigma>n\}}\big] = \mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\Big(\sqrt{\frac{p_2q_1}{p_1q_2}}\Big)^{S_\sigma}\mathbf{1}_{\{\sigma>n\}}\Big],$$

by Lemma 4. The independence property given by Lemma 5 implies that

$$\begin{aligned}
&\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\Big(\sqrt{\frac{p_2q_1}{p_1q_2}}\Big)^{S_\sigma}\mathbf{1}_{\{\sigma>n\}}\Big] \\
&= \mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_1}{p_1q_2}}\Big)^{S_\sigma}\Big]\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\mathbf{1}_{\{\sigma>n\}}\Big] = \frac{\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\mathbf{1}_{\{\sigma>n\}}\Big]}{\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\Big]} \\
&= \frac{\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\big|\sigma > n\Big]}{\mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\Big]}\mathbb{Q}_{p_1}(\sigma > n)
\end{aligned}$$

It follows that

$$\mathbb{Q}_{p_2}(\sigma > n) \leqslant \mathbb{Q}_{p_1}(\sigma > n) \iff \mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\big|\sigma > n\Big] \leqslant \mathbb{E}_{p_1}\Big[\Big(\sqrt{\frac{p_2q_2}{p_1q_1}}\Big)^{\sigma}\Big].$$

Note that $\frac{p_2q_2}{p_1q_1} < 1$, since $p_2 > p_1$. Lemma 3 therefore completes the proof.

□

4 The SDE proof

In this section, we outline the SDE proof of Theorem 1 (ii).

A natural idea is to try to represent the modulus of $B_t^\lambda \triangleq B_t + \lambda t$ as an *Itô process*, namely the solution to an SDE of the form

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt. \tag{1}$$

The reason for doing this is that it allows us to invoke a well-known comparison theorem due to Ikeda and Watanabe [2]. This theorem states essentially that if X^1 and X^2 are diffusions on the same probability space, starting at the same value and each satisfying equations of the form (1) with the same σ but with different drift terms b_1, b_2 with $b_1(x) \leq b_2(x)$ for all x, then $X_t^1 \leq X_t^2$ for all t a.s. (naturally there are conditions on σ and the b's, but we do not worry about them for the moment). This allows us to deduce the result, as the exit time from the symmetric interval in question is simply the hitting time of the corresponding level by the process $|B_t^\lambda|$.

We must first remark, however, that the process $|B_t^\lambda|$ does not satisfy an SDE of the form (1). This is because the semimartingale decomposition of $|B_t^\lambda|$ involves its local time at zero, which is not absolutely continuous with respect to dt (when $\lambda = 0$, Tanaka's formula gives $|B_t| = dW_t + dL_t$ where W_t is a Brownian motion and L_t is the local time of B at zero).

On the other hand, the process $Y_t \triangleq |B_t^\lambda|^2 = (B_t^\lambda)^2$ does satisfy an SDE of the required form, which we shall derive below. The case when $\lambda = 0$ is straightforward; here $Y_t = B_t^2$, and one has

$$dY_t = 2B_t dB_t + dt = 2\sqrt{Y_t} \cdot \frac{B_t}{\sqrt{Y_t}} dB_t + dt = 2\sqrt{Y_t} dW_t + dt, \tag{2}$$

where $W_t \triangleq \int_0^t \frac{B_t}{\sqrt{\rho_t}} dB_t$ is a Brownian motion by Lévy's characterisation. The case $\lambda \neq 0$ requires extra care since the same calculation does not lead to a meaningful SDE for Y_t (the drift part contains B_t^λ which cannot be expressed in terms of Y_t).

We take a Markovian perspective to write down the intrinsic SDE for Y_t. It is well known that the process $|B_t|$ can be equivalently viewed as a Markov process with generator

$$\mathcal{A}f = \frac{1}{2}f''; \ \mathcal{D}(\mathcal{A}) = \left\{f \in C_b^2([0,\infty)) : f'(0+) = 0\right\}.$$

To derive the generator for the process $X_t \triangleq |B_t^\lambda|$, let $x > 0$ be given fixed (it represents the current state $X_t = x$). Explicit calculation shows that

$$\mathbb{P}(B_t^\lambda = x | X_t = x) = \frac{e^{\lambda x}}{e^{\lambda x} + e^{-\lambda x}}, \ \mathbb{P}(B_t^\lambda = -x | X_t = x) = \frac{e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}.$$

Given $X_t = x$, if $B_t^\lambda = x$ the process evolves like $B_s + \lambda s$, while if $B_t^\lambda = -x$ the process evolves like $-B_s - \lambda s$ ($s \in [t, t + \delta t]$). Since the Brownian motion B is symmetric, it is clear that the diffusive part (the second order term) of the generator of X_t is also $\frac{1}{2}\frac{d^2}{dx^2}$. Its drift part (the first order term) is given by

$$\mathbb{P}(B_t^\lambda = x | X_t = x) \times \lambda + \mathbb{P}(B_t^\lambda = -x | X_t = x) \times (-\lambda) = \lambda \tanh \lambda x.$$

In other words, X_t is a Markov process with generator

$$\mathcal{L}^X = \frac{1}{2}\frac{d^2}{dx^2} + \lambda \tanh(\lambda x)\frac{d}{dx}; \ \mathcal{D}(\mathcal{L}^X) = \left\{ f \in C_b^2([0,\infty)) : f'(0+) = 0 \right\}.$$

A simple change of variables $x = \sqrt{y}$ shows that the generator of Y_t is given by

$$\mathcal{L}^Y = 2y\frac{d^2}{dy^2} + \left(1 + 2\lambda\sqrt{y}\tanh(\lambda\sqrt{y})\right)\frac{d}{dy},$$

and the corresponding SDE for Y_t is

$$dY_t = 2\sqrt{Y_t}dW_t + (1 + 2\lambda\sqrt{Y_t}\tanh(\lambda\sqrt{Y_t}))dt. \tag{3}$$

Note that this agrees with (2) when $\lambda = 0$. We now observe that the function

$$\lambda \to 1 + 2\lambda\sqrt{y}\tanh(\lambda\sqrt{y})$$

is increasing for $\lambda \in [0, \infty)$ for all y, and therefore the comparison theorem of Ikeda-Watanabe easily gives our desired result. □

Remark 1. Invoking the theorem of Ikeda-Watanabe requires checking certain conditions on the functions comprising the SDE (3). These conditions are slightly technical to state but straightforward to verify for the SDE in question, so we have chosen not to include them.

Unlike X_t itself, the process Y_t is indeed an Itô diffusion (it does satisfy the SDE (3)) because the local time term will not appear. In fact, in the decomposition

$$dY_t = 2X_t dX_t + dX_t \cdot dX_t, \tag{4}$$

the local time term $X_t dL_t$ (coming from the first term in (4)) vanishes identically due to the fact that L_t only increases when $X_t = 0$ (this is a basic property of the local time).

From general theory, the SDE (3) has a unique strong solution for every initial condition $Y_0 = y \geqslant 0$. One can then show that its solution (with $Y_0 = 0$) has the same distribution as the process $|B_t + \lambda t|^2$. This can be taken as a more direct approach to justify the above considerations.

Remark 2. Theorem 1 seems surprisingly resistant to generalisations. The conclusion is in general not true if the drift is assumed to be non-constant (a simple comparison relation between the drifts does not have a clear implication on the relation between exit times). For instance, if the drift is allowed to depend on the position of the path, one could make a large positive drift when the process goes negative which would push it back to the center, resulting in longer stay in the interval. The theorem also fails to hold in general if the drift is assumed to be time-dependent.

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